Solution to Assignment 2

Supplementary Problems

Note the notations. These problems are valid in all dimensions. Hence we do not use (x, y) to denote a generic point as we do in $\mathbb{R}^2$. Instead, here x or p are used to denote a generic point in $\mathbb{R}^n$.

1. Let S be a non-empty set in $\mathbb{R}^n$. Define its characteristic function χ_S to be $\chi_S(x) = 1$ for $x \in S$ and $\chi_S(x) = 0$ otherwise. Prove the following identities:

   (a) $\chi_{A\cup B} \le \chi_A + \chi_B$.

   (b) $\chi_{A\cup B} = \chi_A + \chi_B$ if and only if $A \bigcap B = \phi$, that is, A and B are disjoint.

   (c) $\chi_{A\cap B} = \chi_A \chi_B$.

   Solution. (a) For $x \in A \cup B, x$ must belong either to A or B. Hence $\chi_{A\cup B}(x) = 1 \le \chi_A(x) + \chi_B(x)$. On the other hand, when x does not belong to $A \cup B$, $\chi_{A\cup B}(x) = 0$ and the inequality clearly holds.

   (b) and (c) are left to you.

2. Let f be integrable in a domain D which satisfies $A \le f \le B$ for two numbers A and B everywhere. Show that
$$A|D| \le \int_D f \le B|D| ,$$
   where $|D|$ is the "area" of D.

   Solution. By assumption, $B - f(x) \ge 0$ for all $x \in D$. Hence
$$\begin{aligned} 0 &\le \int_D (B - f) \\ &= \int_D B - \int_D f \quad \text{(linearity)} \\ &= B|D| - \int_D f , \end{aligned}$$
   and the second inequality follows. The first one can be proved by using $f(x) - A \ge 0$. (The area is better understood as the n-dimensional volume.)

3. Show that a nonnegative, continuous function in a region has zero integral must be the zero function. Does it continue to hold without the continuity assumption?

   Solution. Suppose that the given function is not identically zero. There is some point p_0 lying inside the region D so that $f(p_0)$ is a positive number. By continuity we can find a small ball B centered at p and lying inside D so that $f(p) \ge f(p_0)/2$ for all $p \in B$. Then
$$\begin{aligned} \int_D f &= \int_B f + \int_{D\setminus B} f \quad \text{(decomposition principle)} \\ &\ge \int_B f \quad \text{(positivity)} \\ &\ge \int_B \frac{f(p_0)}{2}\, dA \\ &= \frac{f(p_0)}{2}|B| > 0 , \end{aligned}$$

where $|B|$ is the area of B. Contradiction holds. Hence $f \equiv 0$ in D.

On the other hand, let g be zero in D except $g(q_0) = 1$ at a single $q_0 \in D$. The nonnegative function g is integrable and its integral is equal to zero.

Note. In fact, in real analysis it will be shown that a nonnegative integrable function has zero integral if and only if its zero set is a set of measure zero.